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HW3

A ME 549
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1. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

   (a) Compute the system's transfer function.

   Solution. The system's transfer function is computed via the following relationship

$$G(s) = C(sI - A)^{-1}B + D.$$

   In this instance, $D = 0$ and the other matrices are as follows

$$A = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

   Carrying out the matrix inversion and multiplications yields

$$G(s) = \frac{1}{s+1}.$$

   (b) Is the system BIBO stable?

   Solution. By definition, a system is BIBO stable if the real part of all poles of the system's transfer function lie in the left-half plane. $s = -1$ is the system's only pole, and therefore the system is BIBO stable.

   (c) Is the origin stable in the sense of Lyapunov?

   Solution. I claim that the origin is NOT stable in the sense of Lyapunov.

   Proof. Assume that the origin is stable in the sense of Lyapunov. Then, given any positive-definite matrix N, the equation $A^TP + PA = -N$ has unique solution P that is both positive-definite and symmetric. For simplicity, take $N = I_{2\times 2}$, the identity matrix. Therefore, I should be able to find a P with the above properties, s.t.

$$\begin{bmatrix} -2p_{11} & 10p_{11} \\ 10p_{11} & 20p_{12} + p_{22} \end{bmatrix} = A^TP + PA = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   However, this is a contradiction, since p_{11} cannot simultaneously be $\frac{1}{2}$ and 0 to satisfy the equality. Therefore, $A^TP + PA = -N$ has no solution and the origin is not stable in the sense of Lyapunov. □

2. For the system in Example 3.19, use the Lyapunov function candidate V_2 suggested at the end of the example to

   (a) Show that the origin $(0, 0)$ is uniformly globally asymptotically stable.

   Solution. To show that the origin is uniformly asymptotically stable using the candidate V_2, I need to show that V_2 has the following properties:

   i. $V_2(x) > 0$ for all $x \in C \setminus \{(0, 0)\}$.
   ii. $\dot{V}_2(x) \leq 0$ for all $x \in C$.
   iii. $\dot{V}_2(x) < 0$ for all $x \in C \setminus \{(0, 0)\}$.

   The "global" designation follows immediately from the physical fact that any trajectory, regardless of the initial condition, will settle at the origin due to the dissipative nature of the dynamics of the hybrid system ($\lambda < 1$ constitutes a loss in energy at each collision).

   The candidate function V_2 is defined as:

$$\begin{aligned} V_2(x) &= (1 + \theta \tan^{-1}(x_2))(\frac{1}{2}x_2^2 + \gamma x_1) \\ \theta &= \frac{1 - \lambda^2}{\pi(1 + \lambda^2)} \\ \lambda &= 0.8 \text{ (arbitrarily chosen)} \\ \gamma &= 9.81 \text{ (gravitational constant)} \end{aligned}$$

   For (i), observe that $|\tan^{-1}(x_2)| \leq \frac{\pi}{2}$ for any x_2. So, even at it's most negative, the following holds

$$\theta \tan^{-1}(x_2) \geq -\frac{1 - \lambda^2}{\pi(1 + \lambda^2)} \frac{\pi}{2} = -\frac{1 - \lambda^2}{2(1 + \lambda^2)} \geq -\frac{1}{2}.$$

   Therefore, 1+ this term will always be positive. Since, $x_1 > 0$ by assumption and x_2^2 is necessarily positive, $V_2(x) > 0$ for all $x \neq (0, 0)$.

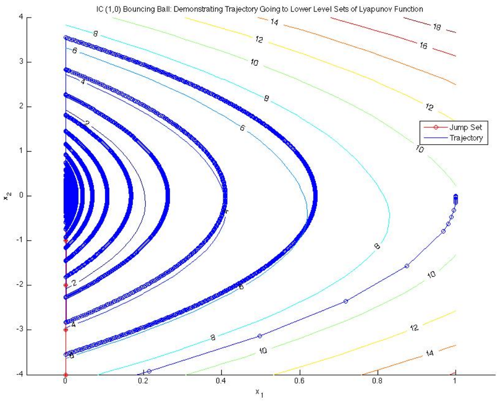
   For (ii), taking the derivative yields

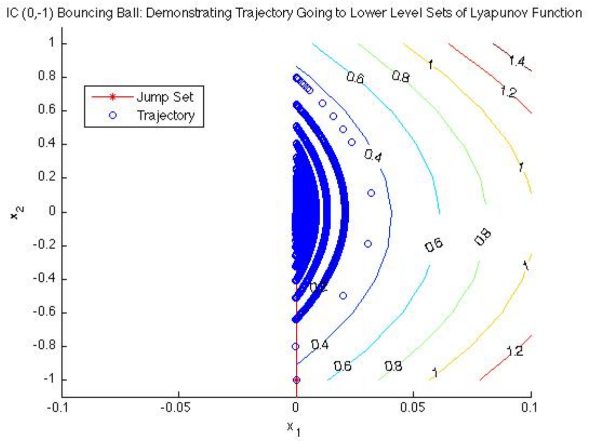
$$\begin{aligned} \dot{V}_2(x) &= \Big(\theta \dot{x}_2 \big(\frac{1}{1 + x_2^2}\big)\Big)\Big(\frac{1}{2}x_2^2 + \gamma x_1\Big) + (1 + \theta \tan^{-1}(x_2))(\dot{x}_2 x_2 + \gamma \dot{x}_1) \\ &= -\Big(\frac{\theta\gamma}{1 + x_2^2}\Big)\Big(\frac{1}{2}x_2^2 + \gamma x_1\Big) + (1 + \theta \tan^{-1}(x_2))(-\gamma x_2 + \gamma x_2) \\ &= -\Big(\frac{\theta\gamma}{1 + x_2^2}\Big)\Big(\frac{1}{2}x_2^2 + \gamma x_1\Big) \leq 0 \text{ for all } x \in C. \end{aligned}$$

   (iii) follows pretty quickly afterwards, $\dot{V}_2(x)$ can only be 0 at the origin. Otherwise, we would require that $\frac{1}{2}x_2^2 = -\gamma x_1$, which would be a contradiction given that $x_1 > 0$ by construction.

(b) In the plane, plot: a) the flow and jump sets, b) level sets of the Lyapunov functions, and c) a solution starting from $(1,0)$ and a solution starting from $(0,-1)$. Show graphically that the motion of the solutions is such that they go from larger to smaller level sets of the Lyapunov function.

Solution. The figures are shown below,





Level sets of the Lyapunov function V_2 are shown as contours with numbers indicating the value of the level set on that contour. It is difficult to see, and I am afraid I was not clever enough to show the direction of the flow in time, however I will try and describe what the plots are illustrating.

The trajectories are converging to $x_1 = 0$ as time evolves due to the dissipative nature ($\lambda < 1$) of the system. Had there been a more exacting choice of contour plotting, I believe that you would see the trajectories in blue lying on level sets of the Lyapunov function. This is more easily seen in the chart on the right. As time evolves, the trajectories fall from a higher level set of V_2 to a lower one at each collision with the ground.

3. Consider the hybrid system with state $x \in \mathbb{R}^2$ and data

$$\begin{array}{rcll} C & = & \{x : x_1 \geq 0\}, & f(x) = \begin{bmatrix} \alpha & \omega \\ -\omega & \alpha \end{bmatrix} x \ \ \forall x \in C \\ D & = & \{x : x_1 = 0, x_2 \leq 0\}, & g(x) = -\gamma x \ \ \forall x \in D, \end{array}$$

where $\gamma > 0$, $\omega > 0$, and $\alpha \in \mathbb{R}$ are the system parameters.

(a) Using the sufficient conditions for Lyapunov stability, find conditions on the system parameters for which the origin of the hybrid system is *uniformly globally pre-asymptotically stable*. Show your work in detail.

Solution. I first need to make use of the following proposition:

Proposition 1. *When the set $\mathcal{A}$ is compact and the Lyapunov function is continuous, the condition (3.2a) (from the notes) iff $V(x) = 0$ for all $x \in \mathcal{A}$, $V(x) > 0$ for all $x \notin \mathcal{A}$, and the values $V(x)$ grow unbounded as $x \in C \cup D \cup G(D)$ grows unbounded.*

In the instance of this problem, $\mathcal{A} = \{(0,0)\}$, which is obviously compact. This proposition comes from the notes and will be used without proof. If I can find a function that satisfies the conditions of the proposition, I get property (3.2a), which would arguably be the hardest to prove, for free. I will pick a candidate function of the following form:

$$V(x) = x^T P x.$$

If P is positive-definite (and, as I would like, symmetric) , I will get $V(x) > 0 \ \forall x \notin \mathcal{A}$, and clearly $V(\mathcal{A}) = 0$. Additionally, V is unbounded as x grows (after all, V grows as a quadratic, which is unbounded). Now, for the gradient condition (3.2b),

$$\begin{aligned} \langle \nabla V, f \rangle = \dot{V}(x) &= x^T(A^T P + PA)x \\ \text{(I want)} &= -x^T N x \end{aligned}$$

for some pos. def. symmetric matrix N. For simplicity's sake take $N = I_{2\times 2}$, the 2×2 identity matrix. From this, I can find conditions on the system parameters α, ω that ensure the Lyapunov conditions will be met. Carrying out the messy algebra,

$$\begin{aligned} p_{11} &= \frac{\alpha - \frac{\omega^2 + 2\alpha^2}{\alpha}}{2\omega^2 + 2\alpha^2} \\ p_{22} &= \frac{-1 - \alpha p_{11}}{\alpha} \\ p_{12} &= \frac{\omega(p_{22} - p_{11})}{2\alpha} \end{aligned}$$

For P to be pos. def.,

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0. \end{aligned}$$

By construction, $\dot{V}(x) = -x^T x$, so a potential candidate for ρ_1 would be $\rho_1(s) = s^2$, where s is the standard Euclidean norm, as $V(x) \leq -\rho_1(s)$ for all $x \in C$. Now, for the last condition (3.2c) (omitting the messy algebra),

$$\begin{aligned} V(g(x)) - V(x)|_{x \in D} &= (\gamma^2 - 1)V(x) \\ &= -(1 - \gamma^2)p_{22}x_2^2 \text{ since } x_1 = 0 \ \forall x \in D \\ &= -(1 - \gamma^2)p_{22}(x_2^2 + x_1^2). \end{aligned}$$

Now, making the substitution of s equals the standard Euclidean norm $\sqrt{x_1^2 + x_2^2}$, we can define the following for the function ρ_2 in the definition of Thm. 3.18.

$$\rho_2(s) = (1 - \gamma^2)p_{22}s^2.$$

For ρ_2 to be pos. def., I need that $1 - \gamma^2 > 0$, which would require that $\gamma \in (-1, 1)$. I would like to flow after a jump, so restrict $\gamma > 0$. Combine the information from the previous two steps to yield the following,

$$\rho(s) = \min(\rho_1(s), \rho_2(s)).$$

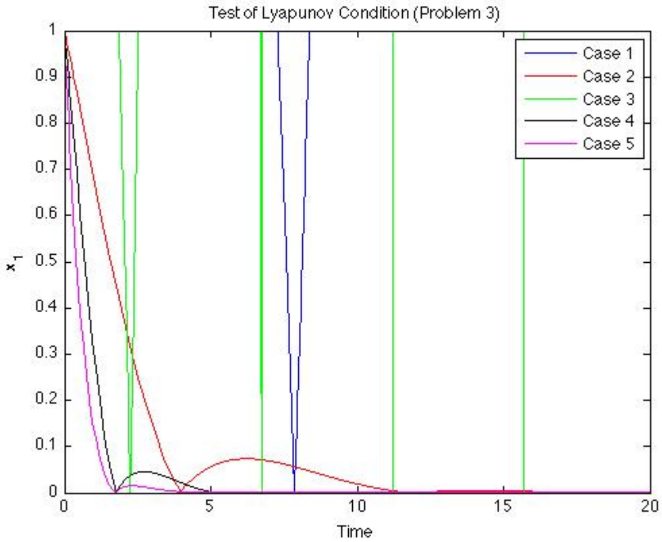
Clearly, ρ is pos. def., as the minimum of two pos. def. functions is also pos. def. Thus, all of the conditions of Thm. 3.18 are satisfied and therefore $\mathcal{A} = \{(0, 0)\}$ is uniformly globally pre-asymptotically stable.

(b) Confirm your answer to item 1 via simulation.

Solution. I am going to test a handful of conditions, just to verify the above. The table below shows what will be tested (all assuming the same initial condition of $x^T = [1\ 0]^T$.)

Case	α	ω	γ	p_{11}	$p_{11}p_{22} - p_{12}^2$
1	0.3	0.2	0.8	−1.7	2.8
2	−0.3	0.4	0.6	1.7	2.8
3	0.7	0.7	0.9	−0.7	0.5
4	−0.7	0.9	0.4	0.7	0.5
5	−1.5	0.9	.99	0.3	0.1

From the table, Cases 1 and 3 should demonstrate instability in the plots. The figure below shows the time series data for the 5 cases.



The figure is in agreement with the data from the table. When the conditions are such that P fails to be pos. def., the time series data demonstrates instability.

(c) Is the origin *uniformly globally asymptotically stable*? Justify your answer.

Solution. For the origin to be uniformly globally asymptotically stable, every maximal solution to $\mathcal{H}$ must also be complete. The flow time domain is unbounded, therefore every maximal solution is also complete. Therefore, the origin is uniformly globally asymptotically stable.

4. For the hybrid system $\mathcal{H}$ that you proposed in Homeworks 1 and 2:

   (a) Define a set $\mathcal{A}$ that is to be stabilized.

   Solution. Begin by redefining the system:

   For simplicity's sake, consider a simple harmonic oscillator with a natural frequency that switches each time the position changes sign. The system is clearly hybrid, since there is a discontinuity in the augmented state (position,velocity, and natural frequency) at each velocity crossing. The behavior will still be oscillatory, but the period of motion will change due to the shift in natural frequency. The flow map will be

$$f(\mathbf{z}) = \begin{pmatrix} z_2 \\ -z_3^2 z_1 \\ 0 \end{pmatrix}$$

   where

$$\mathbf{z} = \begin{pmatrix} x \\ \dot{x} \\ \omega_0 \end{pmatrix}.$$

   This mapping is valid whenever the velocity is not changing signs (i.e. not equal to zero). So, $C = (\mathbb{R} \cap \{0\})^C \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ The jump map could be anything, but let's just say that the natural frequency changes by a factor of γ each time the oscillator passes through the origin ($x_1 = 0$)

$$g(\mathbf{z}) = \begin{pmatrix} z_1 \\ z_2 \\ \gamma z_3 \end{pmatrix}$$

   This mapping is valid for any $\mathbf{z} \in D$, where $D = \{0\} \times \mathbb{R} \times \mathbb{R}_{\geq 0}$. To ensure that the system starts to flow again after the jump map

   Take the initial condition $\omega = \omega_0$, $x = x_0$, and $x_2 = v_0$. Each solution will be of the form

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.$$

   For the set $\mathcal{A}$, observe that the system is a) dissipative for $\gamma > 1$, b) conservative for $\gamma = 1$ and c) gaining energy for $\gamma < 1$. Cases b) and c) are very uninteresting from a stability standpoint, therefore I will focus on a).

   Each time $x_1 = 0$, the amplitude A of the oscillator reduces by a factor of $\frac{1}{\gamma}$. Therefore, for the standard harmonic oscillator whose solution is given above the amplitude (no jumps) is

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$$

Each time $x_1 = 0$, the oscillator starts over with new initial condition defined by

$$\begin{aligned} x_0' &= 0 \\ v_0' &= \frac{v_0}{\gamma^n \omega_0}. \end{aligned}$$

where n represents the n^{th} time the oscillator crossed $x_1 = 0$.
So, as $n \to \infty$, we have that

$$\lim_{n\to\infty} A = \lim_{n\to\infty} \frac{v_0}{\gamma^n \omega_0} = 0.$$

Therefore, the set to stabilize is the origin, $\mathcal{A} = \{(0,0)\}$.

(b) Study the stability properties of the system, either using the definition of UGpAS or using the sufficient conditions in terms of Lyapunov functions.

Solution. This is tricky, as the energy function

$$V(x) = \frac{1}{2}\omega^2 x_1^2 + \frac{1}{2}x_2^2 \text{ assuming } m = 1 \text{ in defn. of SHO}$$

is not continuous in time (and therefore is not continuously differentiable). This isn't the real issue, as the bouncing ball example illustrates, the real issue is that although

$$\alpha_1(s) \leq V(x) \leq \alpha_2(s)$$

where

$$\begin{aligned} \alpha_1(s) &= \min\left(\frac{1}{2}\omega^2 s^2, \frac{1}{2}s^2\right) \\ \alpha_2(s) &= \max\left(\frac{1}{2}\omega^2 s^2, \frac{1}{2}s^2\right) \\ s &= \sqrt{x_1^2 + x_2^2}. \end{aligned}$$

So, uniform stability is established. However, the second and third inequalities will not hold, as both $V(x) = 0$ and $V(g(x)) - V(x) = 0$. So, it seems as though I will have to settle for uniform stability. It seems like there are some modifications that could be made to strengthen the stability conditions on $\mathcal{H}$ however.